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Etim Etim and B. Touschek: A PROPOSAL FOR THE ADMINISTRATION OF RADIATIVE CORRECTIONS. -

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1. INTRODUCTION. -

The necessity of correcting for the emission of real unobserved and the emission and absorption of virtual photons puts the experimenter in a very peculiar dilemma: In order ot compare two experiments carried out with the purpose of studying the same physical process with different experimental arrangements it is necessary to have recourse to a theory - quantum electrodynamics. It is therefore not possible to establish whether or not agreement exists between the two experiments on the basis of the theory of errors alone. The situation can be aggravated if the radiative corrections for the two experiments have been determined by different theorists using different methods of approximation, so that the simple question of agreement of two experiments becomes the subject of multilateral negotiation.

Adone adds urgency to this problem because owing to the high energy and the small mass of the colliding particles, the high energy resolution of the machine itself, as well owing to the fact that in special conditions the currents represented by the in- and outgoing electrons and positrons may add, the radiative corrections may become rather large - up to about 40% in the normal case and more in special configurations.

The present note, which does not pretend to be a contribution to quantum electro-dynamics[1], is a proposal for the rational division of work between the experimenter and the theorist. It will be shown that with an accuracy sufficient to deal with experiments of which the statistical error will for some time hardly be smaller than 5%, it is indeed possible to compare two experiments either directly with one another or with the theoretical predictions based on the calculation of an idealized standard experiment. The direct comparison of two experiments only involves the classical features of electrodynamics and the methods of statistical mechanics can be applied.

The theoretical background for the idea here discussed can be found in the works of Jauch, Rohrlich and Lomon.

2. THE STANDARD EXPERIMENT. -

We shall in the following limit our attention to the reaction

$$e^+ + e^- \rightarrow A^+ + A^- + \Gamma \qquad (1)$$

and we shall assume that $m_A > m$ the electron mass. Γ stands for any number of photons or γ-rays. We shall further only consider the case in which the energy dependence of the reaction (1) calculated in 1st non vanishing approximation is slow. The other extreme - the case of vector meson resonances - will be dealt with in section 5). Process (1) will be considered in the zero momentum system of the electron and positron and we shall assume that the energy of the positron is E (in Adone from about 350 to 1500 MeV). For practical applications one cannot quite forget that what one actually observes in a storage ring is $\Gamma' + e^+ + e^- \rightarrow A^+ + A^- + \Gamma$, but we assume - without proof that the radiation field Γ' accompanying the clashing beams is of sufficiently low density and can therefore he neglected.

To describe the standard experiment we choose an energy $K \ll E$, which we assume to be so large that for all the experiments to which we want to apply our procedure one has $\Delta E < K$ (where ΔE is some average energy resolution) and at the same time so small) that the act of emission of photons with energy $k < K$ can be treated classically by the Bloch Nordsieck[2] method.

The standard experiment then is an observation of the reaction (1) in which Γ consists of an arbitrary number of photons all with energy < K. It must be kept in mind that this standard experiment is somewhat irrealistic, since it would stamp as unobservable also the emission of a large number of photons all of energy K - though of course such processes may carry very little weight in the determination of the cross section.

What matters is that the standard experiments allows one to calculate a finite[3] cross section by means of the standard methods of quantum electrodynamics. We shall call this cross section $d\sigma_t$ and interpret it as the theoretical prediction for the differential cross section of the standard experiment, assuming that A^+ is emitted into an angular range $\sin\theta d\theta d\phi$, where θ is the polar angle referring the direction of flight of the A^+ to that of the e^+ and ϕ is the azimuth. $d\sigma_t$ can be determined by the use of perturbation theory. One can write

(2) $$d\sigma_t = d\sigma_o \left(1 + \sum_{n=1}^{\infty} f_n\right)$$

where $d\sigma_o$ is the cross section obtained in lowest non vanishing order an f_n is proportional to α^n. It has been shown that the leading term in f_n can be assumed to be proportional to

(3) $$f_n = \frac{1}{n!} \left(\beta \log (K/E)\right)^n$$

where β - the "Bond factor", since it assumes the typical value of 0. 07 in Adone is defined by

(4) $$\beta = \frac{\alpha}{(2\pi)^2} \int d\Omega \left(\sum \frac{(pe)}{(pK)} \right)^2 |k|^2$$

Here k is the 4-momentum of a photon ($k^2 = 0$) $d\Omega$ is extended over all dierections of the space part k of k and e the unitvector of polarization, satisfies $(ke) = 0$. p stands for the 4-momenta of all the charged particles created and destroyed in the reaction (1). The sum Σ is extended over all the particles (with a signature +1 for the destruction of a positive and for the creation of a negative particle and with signature -1 otherwise) as well as over the direction of polarization.

It follows from (3) that if K is sufficiently large the convergence of the series (2) will be quite rapid. If (2) is stopped after n=1, the error in the determination of $d\sigma_t$ should therefore not exceed 1.5% for $E/K = 10$. For experiments with an energy resolution of about 5% it can then be concluded that the theorists task could end with the determination of the 1st order radiative correction. This calculation involves the determination of the spectrum of the γ's of bremsstrahlung with energy $K < k < E$ as well as the determination of the interference term between the radiationless matrix-element and the matrix element of lowest order radiative corrections.

3. THE INFRARED CORRECTIONS[4]. -

These corrections fall into the domain of classical physics. They are all that is needed to compare two sufficiently accurate experiments with one another.

The accuracy of any given experimental arrangement designed to measure the reaction (1) can be expressed in terms of a function $\rho(P)$ of a 4-momentum P. This function represents the probability that in the given experiment a 4-momentum loss $\vec{P}, P_o$ (with $P_o > |\vec{P}|$) will remain unobserved $\rho(P)$ is therefore only defined inside and upon the "future" light cone. Since a 4-momentum loss P=0 will with certainty remain unobserved we can assume that

$$\rho(0) = 1 \tag{5}$$

For most practical applications it will suffice to assume that ρ is of the form

$$\rho(P) = \exp(-P_\mu \; E_{\mu\nu} \; P_\nu) \tag{6}$$

in which the exponent is positive definite. The matrix $E_{\mu\nu}$ depends on the details of the experimental arrangement. If for example only energies but no angles are measured one can assume ρ to be of the form $\exp(-P_o^2/2E^2)$, generally, however, ρ will be more complicated. For example in spark chamber experiments the component P_1 i.e. the momentum loss in the direction of flight of the incident electrons and positrons can be measured with high accuracy.

We shall show in generalization of a result previously obtained by Jauch and Rohrlich that the cross section $d\sigma[\rho]$ measured with an apparatus the momentum resolution of which is given by the function ρ can be related to the cross section $d\sigma_t$ by means of

(7) $$d\sigma[\rho] = C[\rho]\, d\sigma_t$$

provided that $d\sigma_t$ shows no violent energy dependence - a case with which we shall deal separately in section 5). $C[\rho]$ is determined entirely by the classical properties of the emission of soft photons and it follows from (7) that two experiments are in agreement if $d\sigma'/C' = d\sigma''/C''$ within the experimental error.

$C[\rho]$ can be calculated by making use of the Bloch Nordsieck theorem. The annihilation of the particles e^+e^- as well as the creation of the particles A^+A^- define a current density $j(x)$, which in general will be different from zero only in a small space time region of dimensions $1/2E$. As long as $K \ll E$ the process of emission of radiation by this current can be treated classically and this allows one to define $\bar{n}_k$ - the average number of photons emitted with momentum k. The Bloch Nordsieck theorem then states that the probability $P(\{n_{\vec{k}}\})$ for emitting n_k photons with momentum k is given by

(8) $$P(\{n_{\vec{k}}\}) = \prod_{\vec{k}} \frac{\bar{n}_{\vec{k}}^{\,n_{\vec{k}}}}{n_{\vec{k}}!}\, e^{-\bar{n}_{\vec{k}}}$$

i.e. by a Poisson distribution. By using standard methods of statistical mechanics and observing that the momentum carried away by the photons is given by $P = \sum_{\vec{k}} n_k k$ (P and k are 4 momenta and $k_o = |\vec{k}|$) it can now be shown that

(9) $$C[\rho] = \int d^4x\; g(x) e^{-h(x)}$$

where $g(x)$ is the Fourier transform of $\rho(P)$, viz.:

(9) $$g(x) = (2\pi)^{-4} \int d^4P\, \rho(P)\, e^{-iPx}$$

and $h(x)$ can be defined in terms of the average number $\bar{n}_{\vec{k}}$ os soft photons:

(10) $$h(x) = \sum \bar{n}_{\vec{k}} (1 - e^{ikx})$$

The sum is extended over all photon momenta satisfying $|\vec{k}| < K$. The fact that (9) only depends on the average number of photons (via h(x)) means that the determination of C is a problem of classical physics, which because of the correspondence principle holds for all the averages of observable quantities.
The numbers $\bar{n}_{\vec{k}}$ are related to the "Bond-factor" defined in equation (4) by

$$\Sigma \bar{n}_{\vec{k}} = \beta \int^{K} \frac{dk}{k} \qquad (11)$$

the divergence of the integral at the lower limit is the infra red divergence. It is clear that h(x) does not show any infrared divergence, since the factor $(1-e^{ikx})$ cuts out the singularity at k=0 of the integrand.

The proof of equation (9) is given by a Fourier transformation of the obvious relation

$$d\sigma = \Sigma \int d^4 P \varrho(P) \mathcal{P}(\{n_{\vec{k}}\}) \delta(\Sigma k n_k - P) \, d\sigma_t \qquad (12)$$

in which the sum is extended over all values of n_k for $|\vec{k}| < K$.

4. SOME SPECIAL CASES AND SIMPLIFICATIONS. -

In the general case the integral in equation (9) is quite complicated. Its evaluation could, however, be entrusted to a computer, which could store the information on h(x) and which would perform the integration for every given ϱ (P) supplied by the experimenter.

The discussion of some simplified problems will show that in many cases one need not go to this extreme and that a fairly good estimate of the infrared corrections can be obtained by considering a few extreme cases.

We first observe that it follows from (4) and (11) that the radiation emitted in process (1) is preferably thrust into 4 cones centred around the direction of flight of the particles $e^{\pm}$ $A^{\pm}$. The width $\Delta\theta$ of the electron cone is $m/E = 1/\gamma$. If also $A^{\pm}$ are relativistic the photons accompanying them will also be confined to a cone of width $m_A/E_A - m_A$. Very often therefore the various cones will not overlap so that the square of the sum in (4) can be replaced by the sum over the squares.

If one neglects the radiative corrections due to the outgoing particles one gets from (4) and (9)

(13) $$\beta = \frac{4\alpha}{\pi v}\left(\frac{1}{2}\log\frac{1+v}{1-v} - v\right) \rightarrow \frac{4\alpha}{\pi}\left(\log 2\gamma - 1\right) \text{(E.R.)}$$

Typical values of β are given in the following table

E(MeV)	250	500	750	1000	1250	1500
β	.048	.055	.058	.061	.063	.065
β_μ		.055	.009	.011	.013	.015

where in the last column we have listed the values of β due to the emission of μ-mesons. For non overlapping cones the total value of β for the case of pair production is represented by the sum of the second and third line. The table shows that the radiative corrections due to the outgoing particles are not negligible - provided that these particles are relativistic, but also that any qualitative conclusions drawn neglecting these outgoing particles should still give a fair picture of the problem.

A remarkable exception to this rule is backward electron positron scattering. In this case there is constructive interference between the outgoing positron and the incoming electron, which raises the Bond factor by a factor four!

Neglecting the corrections due to the outgoing particle we now consider the special case in which the apparatus only resolves the energy but not the momentum of the process. We therefore assume ς to be of the form exp $(-P_o^2/2\,\Delta E^2)$ which gives

(14) $$g(x) = (2\pi)^{-1/2}\,\Delta E\,\delta(\vec{x})\,e^{-1/2\,\Delta E^2 t^2}$$

where we have denoted the fourth coordinate with t and made use of the fact that ς can be arbitrarily extended to negative values of P_o, since its behaviour in that region - owing to the analicity properties of h(x) has no influence on the value of C to be determined from (9).

For h(x) one obtains

(15) $$h(x) = \beta(\log|K\gamma t| - Ci(Kt) + i\,Si\,(Kt))$$

where γ = 1.78 is Euler's constant. For $\Delta E \ll K$ only the asymptotic behaviour of h(x) will be relevant in determining C. In this case one can put Ci(Kt)=0 and Si(Kt)= $\mathcal{E}(t)\,\pi/2$ (with $\mathcal{E}(t)$=1 for t > 0 and =-1 for t < 0). The integral (9) can in this case be directly evaluated and gives

(16) $$C = \pi^{-1/2} \cos\beta \frac{\pi}{2}\, \Gamma(\tfrac{1}{2}-\beta)(\Delta E/\sqrt{2}\,\gamma K)^{\beta}$$

The following table shows some typical values for the correction factor C:

E(MeV)	1	3	10	30	100
C	.76	.82	.89	.97	1.05

(we have assumed a total energy of of 1.5 GeV, K=150 MeV and β = 0.07) The table shows clearly that the approximation used in evaluating the integral breaks down at ΔE = 100 MeV, for one has obviously to expect C < 1. This however is quite logical since the approximation was designed only to hold for $E \ll K$. As ΔE approaches K the integral (9) has to be evaluated using the exact formula (15) for h(x).

A very similar result is obtained for an experiment which is not energy sensitive but measures with an accuracy $\Delta P(\theta)$ the momentum loss in the direction of the colliding electrons and positrons. If this momentum resolution is obtained by measuring the angle between the outgoing particles one has

(17) $$\Delta P(\theta) = \Delta P(0)/|\sin\theta|$$

Provided that $P(\theta) \ll K$ one finds in this case

(18) $$C = \pi^{-1/2}\, \Gamma(\tfrac{1}{2}-\beta)(\Delta P(\theta)/\sqrt{2}\,\gamma K)^{\beta}$$

This equation differs from (16) by the disappearance of the factor $\cos \beta \frac{\pi}{2}$. This is due to the fact that Si cancels under the integration over the angles of the emitted photons.

5. STRONGLY ENERGY DEPENDENT CROSS SECTIONS.-

We consider an extreme case, in which the "theoretical" cross section $d\sigma_t$ can be approximated by a Breit Wigner formula:

$$(19) \qquad d\sigma_t = A\left((E - E_r)^2 + w^2/4\right)^{-1}$$

where E_r is the resonance energy and w is the width of the resonance. We shall deal with the case $w \ll K$ (which is certainly valid for the resonance created in the production of K's by the φ-meson). We assume that the experiment itself has no inherent energy resolution, so that an event is defined by the observation of a pair of K mesons and the absence of photons with energy $> K$. In this case the experimental resolution is entirely defined by the width of the resonance (more precisely by the square root of the sums of the squares of the resonance width and the energy spread of the colliding beams).

The radiative infrared corrections are now obtained by folding the energy dependence (19) into equation (7). The momentum dependence only adds a small correction, which we shall neglect in the following. That it is indeed small can be seen in the following fashion. For a given fixed experimental arrangement the cross section will depend only on $2\sqrt{s} = \sqrt{(p_+ + p_-)^2}$ where p_+ and p_- are respectively the positron and electron 4-momenta. If the 4-momentum P is carried away by photons, then s should be replaced by s', with $2\sqrt{s'} = \sqrt{(p_+ + p_- + P)^2}$ for which we have approximately $\sqrt{s'} = (E - 1/2\, P_o)(1 - P^2/8E^2)$. Neglecting the 2nd term in the second bracket it becomes evident that the measurement of a resonance is the equivalent of a measurement with an energy resolution which is approximately equal to the width of the resonance and with momentum resolution zero. Using the technique of the previous sections and assuming $w \ll K$ it can easily be verified that

$$(20) \qquad \frac{d\sigma}{d\sigma_t} = \left(\frac{w^2 + 4(E-E_r)^2}{K^2 \gamma^2}\right)^{\frac{\beta}{2}} \Gamma\, (1-\beta)\left(\cos\frac{\pi}{2}\beta + \frac{2(E-E_r)}{w} \sin\frac{\pi}{2}\beta\right)$$

The first factor in (20) indicates that the shape of the resonance is slightly modified. The experimental cross section behaves as $((E-E_r)^2 + w^2/4)^{-1+\beta/2}$ it is therefore less peaked and has broader shoulders than the theoretical cross section (19). The Γ-term gives a depression of the whole curve by less than 3% and the last term enhances the high energy branch and depresses the branch below the resonance. The result is a shift "to the right" of the apparent resonance energy $\Delta E_r = (\pi/32)\beta\Gamma$, which is quite small. A more detailed discussion of the experimental situation has been given in this meeting by U. Amaldi, A. De Gasperis and P. Stein.

6. SUMMARY AND CONCLUSIONS. -

Applying the ideas of Lomon to the problem of administering the radiative corrections for the work with Adone we have argued for the adoption of the following procedure.

1) - The theoretical effort should be concentrated on the determination of the first order radiative corrections for a standard experiment in which photons of energy less then $K \simeq 0.1E$ are considered unobservable. The ommission of higher terms in the determination of this cross section involves an error of less than 1.5%.

2) - The completion of the determination of radiative corrections in which detailed account is taken of the special arrangement of every experiment is left to the experimenter, who by the time the first experiments go into the machine should find at his disposal a computer program, ready to digest any experimental situation expressed in terms of a single function $\rho(P)$ of a 4-momentum variable P.

The error which accrues from the second part of the program is due to the inaccuracy of the Bloch Nordsieck method and can be estimated to be $K\beta/E \lesssim 0.7\%$, so that the total error committed by applying the proposed procedure should be less than 2%.

An important feature of the present proposal is the fact that two different experiments with the same process as a theme can be compared even if the theoretical determination of the cross section of the standard process has not yet been carried out.

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